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#THE MATHEMATICS TEACHER

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THE PSYCHOLOGY OF THE EQUATION*

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The equation has two different uses. First, it is an organization of data in such a way as to indicate the operations required to obtain a certain numerical result, this result being the answer to a question which led the worker to frame the equation. So $60 - x = x - 45$ is a good way to organize data to answer the question, "What number is as much less than 60 as it is greater than 45? The equation is here a thing to be *solved.* Second, the equation is the expression of a relation between a variable and one or more other variables. The important thing in this case is to understand the relation or law.

So in $y = kx$, or $y = \dfrac{k}{x}$, or $y = x^2$, or $x^2 + y^2 = k^2$.

In the first case the equation may, of course, represent a special instance of some important relation or law to be understood, and in the second case the equation, then or later, may be solved for some special values of the variables. But in the great majority of cases one or the other purpose is primary, as stated above. The difference is recognized to some extent in the early distinction between (A) organizing numerical data into an equation with x or two equations with x and y, and (B) framing a general formula or equation.

Consider these samples of A:

1. Ten times a certain number is diminished by 6, the result being 36 more than four times the number. What is the number?

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* The investigations on which this article is based were made with the aid of a grant from the Commonwealth Fund.

2. How much water must be added to three pints of a 20% solution of carbolic acid to make it a 5% solution?

3. A man walks at the rate of 3 miles an hour. Two hours after he started another man sets out to overtake him in an automobile going 25 miles an hour. How far will the first man have gone before the second man overtakes him?

No teacher probably expects the pupil to do anything with the resulting equations except solve them. He has to understand certain particular conditions to frame the equation. He does not have to understand the equation as an expression of a general relation or law in order to solve it. Usually he is not expected to.

Consider these samples of *B*:

1. Make a formula or an equation which tells the cost of any number of pounds of starch at 11 cents a pound.

2. Using m, s, and d for minuend, subtrahend and difference, respectively, what equations can you make from them?

3. Using l, w, and h for the inside dimensions of any rectangular tank in inches, write a formula or equation for the cubic capacity of the tank in gallons, counting one gallon as equal to 231 cubic inches.

4. If x is any even number, what is any odd number?

In these cases the pupil is expected only to frame the formula, not to solve it. To frame it he must understand the general relation or law and, to at least a large extent, the formula or equation as its shorthand expression.

The pupil probably realizes a difference between work like that of *A* and work like that of *B*. Also he may be influenced by being given the name *equation* for things like $10x - 6 = 36 - 4x$ or $.05(3 + x) = .20(3)$, and the name *formula* for things like $C = \pi D$, $m - s = d$, $m - d = s$, or $J = \dfrac{lwh}{231}$.

From this point on, however, almost everything that he is taught about equations blurs the distinction. He is given many literal equations like

$$(a-1)x = a^2 - 1,$$

$$\frac{x}{a+b} = a - b,$$

$$a(a-x) = b(b-x),$$

$$\frac{\frac{m+n}{x}}{\frac{1}{m}} = \frac{a}{b}$$

Presumably these represent some important relations between x or y or z and the variables a, b, and c; otherwise they would not be set out as generalized rules. But, in fact, they do not, and all that he is told to do with them is to solve them.

He is taught the coordinate system and set to study $y = x + 4$, $y = x$, $y = x - 4$, and the like. He is much perplexed because hitherto x has always been unknown but only one number when you finally got it known; whereas now you know what it is, but it is 1 or 2 or 3 or 4, or whatever you like. Also, he has been painstakingly learning in simultaneous equations that you can do nothing useful with x and y unless you have two equations, whereas now you cannot have one equation of the new sort without both of them in it.

To the mathematician or logician these may seem to be childish and trivial perplexities. Childish they may be, but since we are teaching children, childish perplexities are precisely the ones we need to prevent. Trivial they certainly are not, at least to the psychologist. For the most uniform and stable connections or bonds that x or y has formed are with "not known," "to be found," "one number and only one right when you find it." The most frequent and emphatic connection or bond that "x and

y both to be dealt with" has formed is with "you must make two equations." The pupil's strongest habits of thought with respect to x and y (with ordinary teaching, *all* his habits of thought with respect to x and y) make the $y = x + 4$ a monstrous perversity. The coordinate system and the facts of the linear equation would in fact be more easily taught to the pupil who has had the customary training with equations to be solved, if NS and EW, were used instead of YY and XX, and if V_1 and V_2 were used instead of Y and X. Almost everything in the usual previous study of equations interferes with the understanding of $y = ax + b$.

Conversely, under the customary methods of teaching, the habit of regarding y as a variable whose value depends on the value attached to x, the habit of shifting x and seeing what happens to y, and the habit of thinking of how y depends on x rather than hastening to solve for something, are likely to interfere with the old solving habits. The pupil who was wont to proceed readily, and even automatically, to solve for any posterior segment of the alphabet that came into view, now hesitates, wonders whether he is to solve it, or graph it, or evaluate it, or perhaps even consider what it and its context mean!

Partly because of a more or less explicit sense of this interference, the majority of teachers and textbooks retain the disturbing y only as a necessary evil to help explain the coordinate system and the graph of an equation, banishing it soon and replacing "the equation $y = 2x + 3$" by "the expression $2x + 3$," and then quietly shifting to "$2x + 3 = 0$," which can be "solved" in peace. But this shift is destructive to the understanding of $y = 2x + 3$ as the expression of a straight-line relation or law whereby one variable always equals 3 more than twice another. This treatment also tends to change the coordinate system from an easy and beautiful organization of what is known about equations into a puzzle to be reconciled with what you do with equations. If this peculiar y always becomes 0 before you do anything to the equation, why bother to learn about y? So works the potent unconscious argument of mental habit.

With quadratics this mysterious appearance and disappearance of y is repeated. All his experiences with equations except

the brief interlude with $y = ax + b$ almost forbid the pupil to do aught with $y = ax^2 + bx + c$ save regard the y as a misprint for 4 or 7. After a renewed exposition of the coordinate system has given him a dawning insight into what such equations mean, the y is spirited away again, and he has equations of the form $ax^2 + bx + c = 0$ to solve. Why he should 'solve' them he probably has not the faintest notion. An additional degree of mystery is added by now calling the values of x the *roots*.

A final element of confusion is introduced by simultaneous quadratics. The y comes back, but now it is (in many modern courses) not a mere second unknown to be solved for, as it was in $3y - 4x = 2$, $y - x = 1$, but is the y of the coordinate system. The two equations are not mere *corpora vilia* for solving, but two real relation-lines, the question being, "Do they cross; if so, at what point or points?"

The algebra of a generation ago was free from this confusion because it did not attempt to teach the equation as the expression of a general relation between a variable and one or more other variables, and did not introduce graphs and the Cartesian coordinates. The equation was a thing to be solved and nothing more.

When teachers of mathematics began to introduce the formula, the concept of a general relation or function, and its graphic treatment, two courses were open to them. They could try to assimilate the new aspect to the old, insisting on the new treatment as if it were only an extension and enrichment of the old. Or they could make a clear distinction, almost a contrast, between the equation as an organization of facts to find some unknown or hidden fact and the equation as an expression of a relation between variables.

The force of the teacher's mental habits and a superficial pedagogy favored the former course, and it was taken. For example, graphs were used as an aid in solving, or in checking solutions of, equations which gave specific numerical values of an unknown. Solving simultaneous quadratics was (and is) taught not as a means of determining the constants in an equation expressing a relation between the variables y and x, but as a means of answering such specific questions as:

1. The sum of the squares of two numbers is 130, and the product of the numbers is 63. Find the numbers.
2. A number is formed by three digits, the third digit being the sum of the other two. The product of the first and third digit exceeds the square of the second by 5. If 396 is added to the number the order of the digits is reversed. Find the number.

In almost every way the new aspect was made to seem as far as possible an outgrowth and extension of the old, or at least a peaceable ally of the old.

A superficial pedagogy might defend this as a case of "apperception," of basing the new idea on familiar ideas, of gradually extending the concept of the equation. A deeper psychology shows that the other course is the one that should have been taken and should be taken now. It appears, in fact, that the two aspects of the equation should be kept distinct from the start and to a large extent throughout; that they should, other things being equal, be given different names, taught at different times and in different ways and with different applications.

TEACHING PARTICULAR EQUATIONS

The equation to be "solved" in order to find some particular quantity should appear early in arithmetic, say in grade 3, in the form

$5 + \ldots = 9$

$3 \times 9 = \ldots$

$24 = \ldots$ 3s

$7 \times \ldots = 21$

$\ldots$ 5s $= 30$

It should be used freely thereafter in all computations where it is the most serviceable form for thought. For example,

$$\frac{3}{4} = \frac{\ldots}{8} \qquad \frac{3}{4} = \ldots \times \frac{1}{2} \qquad \$24 = \ldots \% \text{ of } \$400$$

It should be used as a way of organizing data in the solution of problems where it is a desirable way. Here *Ans.* or ? or "the

number of dollars" may replace the empty space to be filled. This use may be continued in algebra, with such intricacies as are there desirable, but the name for the missing number should under no circumstances be x or y or z. *Ans.* or *Num.* or A or N seem to be the best names ($Ans._1$ and $Ans._2$, etc., being used where two or more equations are framed to state the given facts). Small n (for number) would be a good name, except for the later interference with n for "any number." Q (for question) would be almost as good as Ans. or Num. or N, possibly better. Whenever the equation was used to indicate an exercise in numerical computation or a search for some number, its earmarks would be an equality sign, and Ans. or N or Q. In so far as problems requiring quadratic equations are perpetuated, we shall have $Ans.^2$ or A^2 or N^2 or Q^2, in their solutions.

Such equations are an organization for convenient computation. To frame them rightly requires sagacious handling of the particular facts and relations of the problem situation. To solve them when they are framed requires competent computation. They may be called equations or equalities or arrangements for solving, or even be given no name at all, so far as learning algebra is concerned. Since the number who study arithmetic is enormously greater than the number who study algebra, this type of arrangement should be called an equation.

This work, so far as done during the study of algebra, should be organized under the principle that "Any real question having a discoverable number as its answer can be answered by putting the data together in a suitable equation and solving, providing the data are sufficient to give the answer." Probably it should be completed before the systematic study of the coordinate system, and of linear equations as such, is begun. The questions answered should be in the main genuine ones. Only a few resulting in elaborate fractional equations are necessary, simply to show that, no matter how intricate the relations, they can be handled by the equation technique.

TEACHING GENERAL EQUATIONS

The equation as the expression of a general relation between variables is prepared for in arithmetic in two ways. The idea

of such a general relation has its first stage in the implicit use of rules like "Length in feet = length in inches ÷ 12"; "Cost in cents = (cost in dollars) × 100." In grades 7 and 8 such rules or formulae are more explicitly got in mind, in the case of:—

$C = 2\pi R$ for Number of inches in circumference = 2π(No. of in. in radius),

$I = PRT$ for Number of dollars interest = (Number of dollars in principal) times (the rate in hundredths) times (the time in years),

$H^2 = S_1{}^2 + S_2{}^2$ for the hypotenuse rule, and the like.

The idea of space representation of a relation between two variables becomes familiar in a modern course in arithmetic by reading and making and using as problem matter such graphs as of the growth of a plant, height in relation to age, score in successive practices, population change, rise in costs or wages, and the like.

In algebra the work with formulae will be extended to give training in reading and understanding any formula which expresses correctly any useful relation which could be understood by the pupil in words, in expressing such relations in formulae, and in finding the value of any variable (whose value is worth finding), when the values of the others are given. As a rule, not one such "solving" should be set for any one variable, but many. For example, in Amp. $= \dfrac{volts}{ohms}$ the task will be "How many Amperes"

(a)	when	volts	110	and ohms	22
(b)	"	"	"	"	25
(c)	"	"	220	"	20
(d)	"	"	12	"	2

The use of x and y and z will be avoided in such formulae, as Nunn has advised. The work with graphs will be extended to the comprehension of the Cartesian coordinate system, habituation to y and x as names for the two distances and the understanding of $y = x$, $y = 2x$, $y = \frac{1}{2}x$, $y = x + 2$, $y = x - 2$, $y = \dfrac{1}{x}$, $y = \dfrac{4}{x}$, and other instructive relations: Then comes

the systematic algebra of systematic selected types of equations. These may be called "equations of variables," or "equations of relation lines." This will include the study of $y = cx$ and $y = \frac{c}{x}$ in connection with "varies directly as" and "varies inversely as," of $y = x^2$ and $y = \sqrt{x}$ in connection with tables of squares and square roots and with interpolating in such tables, and of the quadratic equation in general. It seems preferable to use c_1, c_2, c_3, or k_1, k_2, k_3, rather than a, b, and c to express constants in such equations.

The case of solving for x when $y = 0$ will be treated as simply one special case of all the possible solvings. First, y will be found for various assigned values of x, then x will be found for various assigned values of y, including 0, which are specially instructive. Probably graphical solutions, if taught, should come before solutions by computation. Simultaneous equations with x and y will be taught chiefly as a means of answering the question, "Do these relation lines cross? If so where?"—and as a means of answering the question. "If in a linear equation $y = c_1x + c_2$ one point of the curve is 7,4 and another is 13,6 what do c_1 and c_2 equal?"*

In connection with the study of exponents, curves such as $y = x^{\frac{1}{5}}$, $y = x^{\frac{1}{4}}$, $y = x^{\frac{1}{3}}$, $y = x^{\frac{1}{2}}$, $y = x^{\frac{3}{5}}$, $y = x^{\frac{2}{3}}$, $y = x^{\frac{3}{4}}$, $y = x^{\frac{6}{5}}$, $y = x^{\frac{5}{4}}$, $y = x^{\frac{3}{2}}$, $y = x^7$, may be briefly inspected and compared. Some of the practice with logarithms may well be given up to the computations required for plotting a few such curves. The equations $y = x^x$, $(x + c_1)^2 + (y + c_2)^2 = c_3^2$, and others of notable interest may be studied, if time permits. may be briefly inspected and compared. Some of the practice with logarithms may well be given up to the computations required for plotting a few such curves. The equations $y = a^x$, $(x + c_1)^2 + (y + c_2)^2$ c_3^2, and other of notable interest may be studied, if time permits.

Finally the "function" $ax + b$ or the "function" $ax^2 + bx + c$ may be studied as a function without any "$y =$ " to

*Similarly, of course, with $y = c_1x^2 + c_2x + c_3$.

introduce it, but it will then no longer be or bear the name of equation.

As a consequence of this reorganization, the indiscriminate practice with what are now called literal equations would be replaced by two distinct lines of work. First there would be given, in connection with real formulae, practice in expressing any one of the variables in terms of the others, that is, in solving for that variable. Second, there would be given, in connection with typical forms of relation lines, practice in understanding the meaning of the constants concerned as well as the meaning of the two variables.

REACTION VS. RADICALISM IN THE TEACHING OF MATHEMATICS*

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INTRODUCTORY

The writer did not choose his theme but he very readily consents to discuss it, as the price of the privilege to appear on this program. There has not recently been a time in which it was more important to the teaching of mathematics than it is now to get right on mental attitudes towards reform in that teaching. The terms reactionary, conservative, radical, moribund, etc., are being bandied about, sometimes cast as aspersions upon otherwise altogether respectable folk, to such an extent that we need to know how to comport ourselves gracefully when we are labelled for one of the groups. The appearance of such terms in periodical literature may be taken as a pretty sure indication that something in the nature of reform is operating on the minds of the users of the terms. It is readily admitted that no one need be disturbed whether his opponents classify him in the category to which he thinks he belongs, nevertheless it is worth while to know what sort of impression one's attitudes, views and arguments are making on those whose views he seeks to influence.

It is good mathematical practice to be quite clear as to the meanings of the terms we employ. This is also good educational practice, though none too generally followed, as witness the recent discussions on formal discipline, and it is first rate teaching practice.

REACTION AND RADICALISM DEFINED

What do we mean by reaction? When we say a certain member of our profession is reactionary, what is implied or imputed? Webster says reaction is a backward tendency from revolution, reform, or progress. For the present situation we must modify Webster a little. We have had no revolution recently in the teaching of mathematics, however much some public

*An address delivered before the National Council of Teachers of Mathematics, at Chicago, March 1, 1922.

spirits seems to think we need it, and it is in a sense so true as to verge on the trite that we are always in the midst of reform, and always hopeful that out of it some real progress may come. But we are just now seeking, and that too very earnestly, both reform and progress in larger than the usual measure, in the teaching of secondary mathematics. We are probably all agreed that we owe this fortunate fact in the first instance to the lessons of *The World War*, and then to the high services of *The National Committee on Mathematical Requirements* and of *The National Council*.

In view of the way the above mentioned epithets are being used it seems advisable to define both reaction and radicalism as creeds and as movements. Both belief and overt act are of consequence in the present-day problems of reform.

Reaction as a creed, is the disposition to oppose all attempts at reform, in the belief that reform and reformers are always dangerous and generally to be put down.

Reaction as a movement, is resistance to reformative measures and tendencies, either to preserve the existing status, or to rehabilitate a bygone status.

Radicalism as a creed, is the disposition to extreme disregard of past experience in devising and executing reformative measures.

Radicalism as a movement, is the advocacy and practice of extreme measures in reformative movements. "Do the right thing, though the heavens fall," epitomizes the radical procedure, though as practiced, "the right thing" with the radical is apt to mean only what he choses to call right.

The writer has tried to define these terms in the connotation of prevailing usage, rather than as he himself likes to think of them. He prefers to think of the reactionary merely as the one who has an overmastering regard for the past, the Chinese attitude; and of the radical as the one who has an overpowering regard for the future, the Bolshevist attitude. But it is believed that the definitions given above best conform to prevailing meanings, as exemplified by usage.

THE TWO ATTITUDES COMPARED AND CONTRASTED

There is enough of the salt of truth in both extreme attitudes to make them palatable to certain tastes. In our view, however,

we have here two contrasting attitudes toward reform from neither of which high grade achievement can be expected, and neither of which deserves unqualified espousal. It must, however, be borne in mind in employing these terms, that what may seem reactionary or radical at one time may not seem so at another time. For example, it would seem that the educational public is now ready to look upon changes in both matter and method in secondary mathematics as altogether moderate and conservative, that a few years since would have been regarded as distinctly radical. The recommended changes of The National Committee a very few years ago would have been regarded as hopelessly radical, whereas they are now commonly regarded as conservative.

THE CRITERION FOR CREEDS AND MOVEMENTS

The criterion for judging creeds and movements is, however, wholly a social one. Answer the question: "Which creed or movement will best conduce to the welfare of the social whole?" and you have answered the question of which deserves to be espoused. Human welfare is the controlling consideration. Neither radicalism nor reaction can point to high achievement for the race in the past. Nor can we now expect from the extremists any great help on this decidedly human question of reform in the teaching of mathematics. An attitude of mind and a disposition of soul somewhere between the two extremes in question must be looked to in the main, for guidance in the perplexities that confront us in our present reform.

CONDITIONS TO SOUND ADVANCE

There are two indispensable conditions to sound advance in every important human movement, namely: First, there must be due consideration given to the lessons of past experience, and second, there must be a conviction of the need or supreme desirability of the proposed movement and a willingness to give an unprejudiced hearing and a fair test to new and untried, or insufficiently tried, ideas, methods and theories. Experience must be summarized and preserved. Here comes the function of the conservative, who ought to be a librarian; not a teacher. But experience must also be enriched. It is to be

preserved only that it may be used and extended. Here comes in the function of the reformer and the reorganizer. The conditions of sound advance call very feebly for the radical and still less for the reactionary.

THE RIGHT USE OF EXPERIENCE

It is worth noting in passing that experience, rightly used, is not a headlight nor a taillight so much as it is a boulevard light, throwing its rays to all points of the compass and the sky. The reactionary, often styling himself conservative, interests himself mainly in the illuminated region that lies behind the car of progress; if he is an extremist the region of interest lies far behind. The radical is interested only in the dimly lighted region far ahead, and has no serious regard for the region behind. The reactionary sees in Patrick Henry's famous utterance: "I have but one lamp by which my feet are guided, and that is the lamp of experience," the function of the taillight only. The radical sees only the function of the headlight and, indeed, he doesn't care much for the lamp at all. He prefers adventure to carefully guided exploration, choosing to leap before rather than after he looks, and enjoying the exhilaration of the hazard. The reactionary eschews hazard and chooses only time-tried and travelworn routes. He is apt to be a confirmed routinist.

Discoveries have been hit upon by a bold plunge into the dark, and the radical enjoys risking the improbable, while the reactionary will have nothing of risk and advocates "letting well enough alone," without greatly troubling about whether it really is well enough. The reactionary allows "The bird in the hand is worth two in the bush," and the radical allows the bird in hand to escape for the chance of those in the bush. To the radical the goal glows with blinding intensity, while with the reactionary the point of departure is the goal. They do very poor team-work when harnessed together as we should like to see all lovers of progress, but they are alike in some respects. They are alike in giving slight regard to ways and means of attaining the goal. They are alike also in that both use the present status as the fulcrum for the levers of their thinking, and again in that they are without purchase and powerless so

soon as the existing status has passed. Neither cares to construct, and neither can construct. The radical spurns the existing status and the reactionary tolerates it only as an alternative to reform. The war profiteer who wants the war status perpetuated is the reactionary, and the political demagogue who would force the return of the normal status by violence is the radical, if we may draw upon current history for an illustration. Neither extremist is helping appreciably on our present problem of reconstruction. The radical is dangerous, but the reactionary is well-nigh hopeless. If the alternative were to choose affiliation with one camp or the other, the writer at any rate would prefer that of radicalism to reaction.

THE CORRECT ATTITUDE

Fortunately, however, there is a middle ground between the extremes that promises with some historic guaranties, to do more than either extreme for sound progress. Patrick Henry followed the remark quoted above with: "... and judging by the past, I wish to know, etc.," certain things that pertained to the present and the future. What he was able to gather from the past was now to be employed to throw light upon the uncertainties and probabilities that lay about and ahead of him. This the reactionary would regard as useless and the radical would regard it as unnecessary. The open-minded searcher for guidance in establishing justice regarded it as the only hope for success in his search. The middle ground between the extremes of radicalism and reaction is *openmindedness.* We need today the type of openmindedness of Paul of Tarsus, who counseled: "Try all things, and hold fast that which is good." Out of this attitude of mind all past progress has come and to it we must look for all future progress. It is as germane to the teaching of secondary mathematics and in fact to the teaching of any science whatsoever as it was to the sorely perplexed disciples for whom it was indited.

TWO SIGNIFICANT REPORTS

The writer suspects that whoever suggested the theme of this paper has been in close contact with the situation that now environs the teaching of secondary mathematics, and has met

and felt some of the states of mind that are likely to jeopardize the reforms that many of the best friends of improvement are seeking to inaugurate. There lie before the teachers of secondary mathematics two highly significant reports with recommended reforms, and the vital question is what the teachers will do with the recommendations. We are of course alluding to the report of *The Commission of The N. E. A.* on the reorganization of secondary mathematics, *Bulletin No. 1, 1920, Bureau of Education, Washington, D. C.*, and the report of *The National Committee on Mathematical Requirements,* which is entitled: *The Reorganization of The First Courses in Secondary School Mathematics, Secondary School Circular No. 5, Bureau of Education, Washington, D. C.* One may very well wish that the recommendations of the two consequential bodies concerned in these reports might have been in closer agreement on essentials, to the end that those who attend to such reports and try to profit by them might be a little less bewildered by the conflicting proposals of the two reports. In the writer's opinion the total effect of the two reports will be less influential on teaching practice than would either alone, by reason of the obvious divergence of views and purposes displayed by the reports. Considering that *The N. E. A. Commission* knew before publishing their own report, of *The National Committee's* report and of the wide divergences and conflicting recommendations of their own reports, one cannot but wonder whether there was as sincere a desire to help the situation for secondary mathematics, as there was to nullify the probable effects of the report of *The National Committee.* The writer, moreover, is not alone in the possession of this feeling. Still *The N. E. A. Commission's* report may be regarded as envisaging the problem of improvement from the point of view of administrative officials and *The National Committee's* report as viewing the problem from the standpoint of teachers of mathematics and mathematical experts. Perhaps both viewpoints are needed and, if so, it is well that both should be set forth so explicitly as they are in the formal reports.

THE IMPERATIVE NEED AT PRESENT

Lists of recommendations can amount to nothing more than an exhibit of what a dozen or so people can agree upon as de-

sirable innovations, until enough teachers of neither the reactionary nor radical sort, but of the openminded type, put these recommendations to the practical tests of the classroom in sufficiently numerous instances and under sufficiently varied conditions to establish their general applicability and suitability. Even more than this is imperative. The results of these tryouts, which should be carried to an adequate stage of finality and conclusiveness, must be given wide publicity. The individual reports should be collated, digested and evaluated, and the final conclusions be made generally accessible. This pragmatic test of trial under widely varied conditions, of the proposals of *The National Committee* is a professional duty resting upon the openminded element of the profession, a duty that is so supremely opportune and offers opportunities so superb for high professional service, that capable workers ought not to be lacking. This is no task for the reactionary nor the extreme radical. Serious, thoughtful, consistent, consecutive, deliberate, unsensational, and so forth, are the adjectives that should qualify the performances called for here. Participants in such an undertaking would deserve a degree of consideration no less than belongs to benefactors of the age. Such work would mark an era of advance in the teaching of secondary mathematics as beginning in this year of Grace 1922.

A CONCRETE PROPOSAL

To be very concrete just now the writer believes that if brief expository papers on single class exercises along lines of the proposals of *The National Committee* could appear as a regular feature of *The Mathematics Teacher*—and was this journal not started to promote the work of *The National Committee?*—it would capitalize the Committee's proposals, stimulate and give nation-wide tone to teaching, and incidentally, would greatly extend the subscription list of this already very helpful journal. Urging, as he is, the promotion of the evaluation of the proposals of *The National Committee*, perhaps he ought to say that he is not and has not been a member of this committee, and that he has no partisan interest in its success or failure. He claims only the credit of being a modest worker for improvement in the field of mathematical pedagogy, and he believes that

great promise of immediate improvement has been prepared by this committee and feels that it would be little short of a calamity to allow the opportunity to take a forward step to pass unused, as it surely will if the proposals of the report are not followed up with reasonable promptness.

NEED FOR OPENMINDED PROGRESSIVES

To overcome the inertia of school routine, including habituated administrative procedures, habituated teaching practice and teacher-training, and habituated fear of college entrance obstacles, enhanced by the vested interests of numerous authors and powerful publishing houses, will require for a considerable period a devoted body of some numerical strength of openminded, energetic and progressive teachers. Mere *ex cathedra* pronouncements and untested opinions as to what constitutes improvement, even if backed by *uncritical* classroom experience, are of little value now-a-days. Subjective judgments and uncriticized experiential routine have no standing in the courts of the reform that is now making. Impersonal, objective evidence alone can carry conviction. Let us furnish it to the journals that want to aid us in our projects.

THE RADICAL AND THE REACTIONARY OF USE, AFTER ALL

While the openminded attitude and an abiding faith in better attainable results are particularly needed and are the only safe guaranty of sound advance, still pending the appearance of the requisite literature for aiding reorganization and reform, we shall need the point of view of the radical. We shall need it to prevent us from dropping too easily into the old ruts when the pedagogic bogies of ridicule and loss of position leer at the reformer. Let us remember there is the *earnest* radical and also just the *agitating* radical. The former seeks to justify his reformative proposals and attempts, and he may well be heard with attention. He even sees a vision sometimes that he can impart to others who are more capable of realizing it than he is. Who shall say that in the midst of the present unrest we do not need visions? The earnest radical is needed now as always. We may heartily bid him God-speed! The agitating radical cares only to agitate for the resistance and overthrow of the

existing order, regardless of whether there is an available substitute. He doesn't seem to care so much actually to resist and overthrow, as to agitate for them. For the sake of avoiding mistaking the useful for the useless variety of radical, we may well tolerate even the agitating radical. The existing order is, as we have seen, pretty well buttressed, and it is stimulating exercise at least, to winnow the tares from the good grains of doctrine and doctrinaires.

What now shall be regarded as the professional attitude toward the reactionary? The writer believes that he also should be given a hearing in our councils, for sound reform must grow out of the existing status. The reactionary, particularly if he is of the self-styled conservative type, can and will keep us reminded where we are, whence we came, and that something after all has been attained in mathematics teaching that is worth preserving and, perhaps, improving. We shall gain nothing by losing our moorings, and the reactionary will help us avoid this mistake. The writer seems reaching the conclusion that under present conditions both camps of extremists should be kept in our ranks and be given attentive hearings. Both types of extremist are at least friends of some sort of mathematical education for youth, and it will be more profitable to mathematical culture to reserve our ammunition for the foes of such culture than to spend it on factions of our own camp. The radical may catch gleams of the glorious tomorrow, the inspiration of which in the midst of the benumbing detail of experimental evaluation even the openminded and the progressives need, and the reactionary may enable all to see a little more attractively the splendid results of the past.

OUR PROFESSIONAL DUTY

The rank and file of us will however be of most use to the age by prosaic experimentation, modification and adaptation, experimentation again, and so on, each in his own way in the work of reorganization and of solid reform, all being guided by the best available consensus of the routes that will lead most directly to better things in the teaching of mathematics in our secondary schools. Let the radical who must be that or nothing be radical, for what we need is energy. The radical attitude is more help-

ful than that of the satisfied and complacent routinist. Let the reactionary who must be that or nothing be reactionary, for action is needed, and if he be a good sample, he is more likely to see the light of reform than is the complacent routinist. It takes energy to be either a reactionary or a radical, and energy will at least loosen things. Good counsel to the professional brethren would seem to be as follows: "Be an open-minded worker for reform if you can. If you cannot be this, be a radical. If you can be neither, be a reactionary." The one who cannot be any of these is in grave danger of gravitating into a complacent routinist, and thus become the least helpful of all. He may be inconspicuous and comfortable while his more courageous brethren are bearing the burden and the buffetings that his very type of conduct has brought on.

CONCLUSION

May every one of us aspire to be a *force* in the reform now making, positive if we can be, negative if we must be, but still an *active force*. No real teacher dares look on the struggle with indifference. To do so means loss of professional standing. The stake in the game now going on is nothing less than the alternative of a substantial reduction of the mathematical element in education to the vanishing point, or the marked improvement of quality of output of that element in secondary schools. Can we not forego the privilege of bandying epithets for a season and roll up our sleeves for reorganization and reform, just now along the lines indicated by *The National Committee*, whose mantle in the nature of things is soon to fall upon *The National Council?* LET'S GO!

THE DEFINITION OF SIMILARITY

By GEORGE W. EVANS
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I suggest for the teaching of elementary geometry a change in the customary definition of similar figures. The altered definition will not only be more generally applicable, more nearly commensurate with the idea which we all have of similarity, but will tend towards simplifying and unifying the teaching of certain separate parts of the subject of elementary geometry.

No change is proposed in the definition of similar triangles; leave that as it is. The change suggested is to define similar figures in general, as follows:

Two figures are similar when any three points in one form a triangle similar to the triangle formed by the three corresponding points in the other.

Probably every student of geometry even now gets an adequate idea of similarity, at least in plane figures; maps and plans are accepted at once as similar, corresponding lines are intuitively selected, nobody raises a whisper of objection when it is pointed out that the ratio of the areas of two similar figures, no matter how irregularly curved the boundaries, is the square of the ratio of two corresponding lines. But there is a gap in the argument. We have a definition of similar triangles, and of similar polygons; circles we prudently ignore, as we do all curvilinear figures; and with solids we are apparently helpless, for we confine ourselves to pointing out relations between ratios of certain pet lines, like radii, in figures of certain specified types, like spheres, or cylinders generated by the revolution of similar rectangles.* We may hope that the student concludes that there is a very important idea, applicable to all types of figures; but this idea we fail to define with any generality or to apply with scientific thoroughness.

It is true that some teachers make use of what they call perspective,** or homothetic† position, for defining similarity of

* There is a cumbrous definition for similar polyhedra: See Phillips & Fisher, Elem. Geom. § 714.

** Young & Jackson, Plane Geometry, p. 179.

† Hadamard, Geometrie Elementaire, chapitre V.

plane figures; and that this may be extended to solid figures. This must seem to the high school pupil somewhat remote from the methods of constructing or of analyzing the figures concerned, as may be seen in trying to show that two maps of the State of New York may be placed homothetically, and that then points of the same name would be in homothetic correspondence. Two models of the same solid figure would be even more perplexing. The one luminous case of advantage in teaching is that of parallel sections of a pyramid, and of course we are not throwing that advantage away. Against the general use of such a definition and method, however, is the fact that similarity belongs to the figures themselves, and not to their relative position.

The definition of similar figures here proposed is not really new. It is substantially the same as saying that similar figures are distinguished from each other by a change of scale. As to the question of selecting corresponding points, it is best to consider first how we construct, by points, a figure similar to a given figure. Let the given figure be $A\ B\ C\ D$ . . . . . , the similar figure to which shall be named $A'\ B'\ C'\ D'$ . . . . . ; and the distance AB be x. Take any two points at a distance x', calling either A' and the other B'. Construct $A'\ B'\ C'$ on $A\ B\ C$, $A'\ B'\ D'$ *on* $A\ B\ D$, and so on. In these two figures any two corresponding lines (lines joining corresponding points) have the ratio $x' : x$.

If two figures are given similar, we must have some rule for selecting the first two pairs of corresponding points; and we shall need to know whether the rotational order in the two given figures is the same or not. Other corresponding points can now be picked out by constructing similar triangles. The following theorems are obvious and convenient.

Points that divide corresponding distances in the same ratio are corresponding points.

Lines that divide corresponding angles in the same ratio are corresponding lines.

Lines that pass through corresponding points and make equal positive angles with corresponding lines are corresponding lines.

The intersections of corresponding lines are corresponding points.

If, instead of having the similarity presupposed, we have two figures whose similarity is in question, we shall need a rule for selecting corresponding points. Wherever such a need arises in elementary geometry, there is a rule of correspondence, and the rule is in general more complicated in statement than in application. Thus, in similar triangles, the angles correspond according to their size, and the sides and vertices as related to them; or the sides correspond according to relative length, and their intersections and angles as related to them. In circles we may select any point whatever on one circumference to correspond with any point whatever on the other, and succeeding points according to their interval in degrees from these two first points. The centres always correspond, and the rotational order of corresponding points is either the same or opposite, in the two circles, for all points. It is necessary that pupils should know how to select corresponding points, but no one, of course, would advocate the learning by heart of such rules as those here given. One learns these rules generally in the same way that one learns the rules for personal conduct or for equilibrium in walking. They are necessary, whatever definition of similarity is adopted.

The investigation of circles as similar figures requires a study of the triangles belonging to a circle. Such are central triangles (made by two radii and the chord joining their ends), inscribed triangles, circumscribed triangles, and others. For this study we develop the theorems about central angles, inscribed angles, angles of tangent and chord, tangent and secant, two chords, two secants, two tangents (circumscribed angle). Even without most of the theorems just referred to, we can prove that circles are similar figures, and that arcs of the same number of degrees are similar figures; but the theorems are very convenient; for example, in studying regular polygons.

From this point of view, the subject of angles connected with a circle, instead of being a digression as in the usual treatment, becomes contributory to the general idea of similarity.

It is an obvious consequence of our definition that "pairs of similar triangles, similarly put together, give similar figures; and every pair of similar figures (polygons) is composed of

* DeMorgan, quoted in Heath's Euclid, vol. II, p. 232.

similar triangles similarly put together"*; and it is consequently proved that the ratio of the areas of similar polygons is the square of the ratio of similitude, as well as that the ratio of the perimeters is the same as the ratio of similitude. The analogous theorems about similar polyedra are directly obtained in the same way, that about volumes depending on the study of a tetraedron.

When we come to similar curvilinear figures, the areas being determined as the limits of the areas of variable polygons, the proposed definition is of immediate advantage. For the variable polygons in the two figures are determined by corresponding points, are consequently similar, and have as ratio of areas the square of the constant ratio of similitude. Their limits therefore have the same ratio. There is no reason why this conclusion, even in elementary geometry, should be confined to circles; it must be remembered, however, that proving the number $\frac{S}{r^2}$ to be constant is a different thing from determining its value.

Let us suppose, for example, that we have two convex, irregularly curved, but similar, closed contours, the areas enclosed by which are to be compared. Within either contour draw as many square units (inches, say) as possible, contiguous and completely included by the contour. Then, in addition, draw, in the same way, as many squares as possible, one-tenth of an inch on a side, one hundredth of an inch in area; then the next smaller subdivision (one ten-thousandth of an inch in area), and so on. There will result a polygon less than the area enclosed by the contour. By annexing to this polygon, at any completed stage of its development, additional squares of the smallest size appearing in it, another polygon is derived which completely encloses the area of the figure.

If now any rectangle be drawn around the contour with its sides parallel to the sides of the enclosed squares, each square of the super-added set will have a place to itself on one side of the rectangle, reached by sliding on a line perpendicular thereto. The inner polygon, then, without the super-added squares, will differ from the area enclosed in the contour by less than the

area of a complete row of these smallest squares arranged along the four sides of the rectangle. That is, the polygon differs from the area in question by less than $\frac{p}{10^n}$, where p is constant, and n may be (theoretically) increased as much as we like. The inner polygon, then, approaches the required area as its limit.

Within the other contour another inner polygon can be constructed, similar to that within the first contour, by locating for vertices the points corresponding to the vertices of the first inner polygon. The areas of the polygons will have for ratio the square of the ratio of similitude of the two given contours. Since the areas inclosed by the contours are respectively the limits of the areas of the inner polygons, the ratio of the areas inclosed by the contours is the square of the ratio of similitude.

Figures with concavities in their boundaries can of course be dissected into convex figures.

An obvious extension of this argument proves that similar solids, whatever their boundaries, have for ratio of volumes the cube of the ratio of similitude.

The rectangular figures just described have variable boundaries, but the limits of those boundaries are, of course, *not* the boundaries of the areas or volumes measured.

If now we define the length of a convex contour as the limit of the length of the perimeter of an inscribed convex polygon, the inscribed polygons in two similar contours having for vertices corresponding points; and if we define the area of a curved convex bounding surface as the limit of the area of an inscribed convex polyedron, we have complete command of the important numerical relations due to the similarity of any sort of figures, solid or plane. And this command does not depend upon hazily conceived analogies, or upon the authority of the learned teacher.

Those who have been reading the reports of the National Committee on Secondary Mathematics, especially that on the functional idea, will perhaps see in this definition a further advantage, in that it emphasizes correspondences of numbers and of geometrical entities.

THE PLACE OF ELEMENTARY CALCULUS IN SENIOR HIGH SCHOOL MATHEMATICS

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Various articles which have appeared in THE MATHEMATICS TEACHER during the past year relative to the present-day status of mathematics in our secondary schools and in the first part of college work show that the mathematics curriculum is in a state of transition. These articles point out that in secondary mathematics the aim is to make the subject contribute as much as possible "to a better appreciation of civilization and a better understanding of life about us." Another and more striking modification is the new principle stated in the Report on College Entrance. It says that the preparation in mathematics should be the kind that will be the best basis for work in all the main departments in college rather than that the preparation should show merely the mastery of certain standard parts of abstract mathematics. The Report of the National Committee on Mathematical Requirements is especially interesting in that it recommends as electives in senior high school mathematics simple courses in elementary calculus and in the history of mathematics. This is in harmony with the trend in the status of mathematics as I found it apparently to be from a year's graduate study which involved an investigation also along this line. I shall put some of the results of my investigation in the form of conclusions in the hope that they may be of interest to the reader and may aid those in doubt to decide in favor of these new departures in the mathematics curriculum.

I. *The Status of Mathematics in Foreign Schools Corresponding to our High Schools.*

The main types of the material used as a basis for this investigation were the leading European textbooks on elementary calculus which have been written for use in their secondary schools, the class lists or programs of study of certain particular schools as the Manchester Grammar School of England and the Lycées of France, and a comparison of the curricula in mathematics as found in different countries.

In at least ten of these foreign countries, calculus is taught before the end of the year which corresponds to our twelfth school year to pupils whose ages range from fifteen to eighteen years. It also is apparent that the United States is far behind in the work that is being done in mathematics in our secondary schools. One of the main reasons why the pupils in the schools abroad are able to accomplish this work in mathematics which throws them two years ahead of our pupils by the end of the twelfth school year is because of the more profitable arrangement of the curricula in mathematics in their schools. A more profitable arrangement has already been made in the mathematics curriculum of our standard junior high school and an attempt is now being made to do a similar thing for the mathematics curriculum of our senior high school.

II. *The Trend of Mathematics in our Public School System.* In colonial times, in the schools corresponding approximately to our high schools, the arithmetic consisted of operations with integral numbers and of some work in fractions and proportion. A few decades later, when our high schools first began to make their appearance, arithmetic had become more difficult and included such topics as stocks and exchange, circulating decimals, alligation, cube root, and permutations and combinations. Then it was realized that it would be better not to teach so much of the more difficult parts of arithmetic but more of the simpler parts of some of the higher branches in mathematics. Hence, during the latter part of the nineteenth century, algebra and geometry became a part of the high school mathematics, and these were followed a few years later by the introduction of trigonometry.

We are again passing through a transition period in the mathematical content of our curricula analogous to the one just described. According to the arrangement of the work in mathematics in the standard junior high school, propaedeutic courses are given in algebra, geometry, and trigonometry by the end of the ninth school year. The nature of the subject-matter in these three courses differs from the traditional textbook type. Instead of an unnecessary amount of time and space being given to puzzle problems, problems of mere complexity, and problems of a too advanced character merely to make the work exhaustive,

the time and space are devoted to a simple, careful, and helpful introduction into these subjects.

This preparation in mathematics in the junior high school makes it possible to complete all the desirable present high school mathematics by the end of the eleventh school year. Since this arrangement leaves room for an elective course in elementary calculus in the twelfth school year, the question arises, is this subject of sufficient importance?

III. *The Important Position Occupied by Calculus in the Mathematics Structure.*

Those who are familiar with the courses in engineering know that calculus furnishes so many of the fundamental principles in the theory of engineering that much of the work in this field would be impossible without calculus. A similar situation exists in the fields of physics and mechanics. Calculus also plays an important part in the mathematics of electricity, light, shipbuilding, aviation, wireless telegraphy, the chemistry of heat, and the theory of probability in statistics. In the field of pure mathematics, calculus is used extensively in differential equations, integral equations, the theory of the functions of a complex variable, and the theory of relativity. In brief, calculus is one of the most important aids in applied mathematics and contributes largely to the field of pure mathematics. Thus calculus forms *the* connecting link between the fields of applied and pure mathematics *and* the school work in the reorganized mathematics preceding the last year in the senior high school. Therefore, because of the important position occupied by calculus in the mathematics structure, twelfth year pupils who are interested in mathematics should be given the opportunity to become acquainted with the subject provided it is not too difficult.

IV. *A Historical Survey of the Natural Growth of Calculus in the Development of Mathematics.*

In this survey the chronological order was followed. The first general step in the development of calculus is the method of exhaustion and was used as early as 400 B. C. The remaining general steps originated in the seventeenth century, and the first textbook on differential calculus was published during this

century (1696). After Cauchy had given the first rigorous deduction of the derivative as a limit (c. 1820), calculus began to be rapidly extended in an increasing number of ways in both applied and pure mathematics. This study of the primitive methods in calculus shows that a first course in elementary calculus lies well within the mental grasp of twelfth year pupils in mathematics.

V. *A Comparison of the Leading Textbooks on Elementary Calculus for Beginners and for Self-Instruction.*

The English authors lead in this type of book. They keep in mind the needs of the learner and they keep the theory closely allied with work in physics and mechanics. There are also a number of German textbooks of this type, but the German authors adhere more closely to the scientific method and include too much work of a purely abstract nature. The French combine their work in elementary calculus with that of their advanced courses in algebra. There are almost no American textbooks on elementary calculus which have been written for others than college students. This comparison of textbooks shows that a number of the best teachers of secondary mathematics do not consider elementary calculus too difficult for their pupils.

VI. *The Trend of American Education in General.*

The trend of American education in general shows that the length of the school term has increased from that of a few months in the early days of the public school system to a term of nine or ten months with compulsory education under certain conditions. The number of years that the average child attends school has likewise increased. It is also true that many pupils besides those who go to college do not cease their educational activity at the close of their public school life. This is indicated by the great number of people taking correspondence courses and extension courses, and by the great number of evening schools and shop schools. The success of chautauquas also shows that the public is still athirst for knowledge. Indeed, education can no longer be thought of so much as a preparation for life but rather as a part of life itself for many years. However, after the public school period, the person who studies along any particular line no longer has the benefit of the teacher's

guidance. Hence, those who wish to study further along the line of mathematics will find desirable a preparatory knowledge of differentiation and integration such as would be given in a course in elementary calculus in the senior high school.

VII. *General Conclusions.*

The study of this problem also shows that the proposed course in elementary calculus is useful; that it generates an appreciation of mathematics; that it develops the kind of thinking that can be applied to other situations in life; that it is interesting to the pupils who elect mathematics; and that it is not too difficult for such pupils.

The theory and applications of elementary calculus can be so simplified that it becomes an interesting and extremely valuable subject for such pupils in the senior high school who show fair ability in mathematics.

It is evident that a course in elementary calculus should be included in the *elective* mathematics of the senior high school for the following reasons:

It is necessary if our pupils are to have an equal opportunity in mathematics with pupils in schools abroad;

Such a course is needed to satisfy the demand for additional mathematics in the senior high school which has been created by the trend of mathematics in our public school system;

Calculus is one of the most important aids in applied mathematics and contributes largely to the field of pure mathematics;

Such a course will enable the pupil who intends to enter any field of applied mathematics to continue further his study of calculus as found in such a field without the aid of a teacher.

EXPERIMENTAL GEOMETRY*

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Before beginning a discussion of the actual topic assigned for this paper, let us consider for a short time what are the requirements that determine whether or not a subject should have a place in the school curriculum. It does seem that we have emphasized two things entirely too much. These are, the ease of handling the subject on the part of pupil and teacher and the practical value of the subject. These two should be considered of course, but there are other considerations of more importance.

The two main objectives of school work are the development of character and the training of the mind and any subject which does not assist in these two objectives better than any other which may be substituted for it is unworthy a place in the course of study of any school. We will except, however, such subjects as stenography, printing, etc., when pursued for the purpose of preparing the individual for a definite occupation. A third objective is the practical value of the subject. This must not be overlooked. If we can give the pupil some experiences in school similar to those that he will encounter in later life, we are gaining much. But on the other hand, there is danger of overemphasizing this third objective. Even Manual Training when taught in the usual way has more value in the development of character and in formal discipline than it has as practical worth. The boy who learns to make a useful article at the manual training bench will cultivate habits of neatness and thoroughness that will enable him to do better work elsewhere.

Now, how does Mathematics stand the test of the preceding paragraphs? In the first place it is a study of the unchangeable laws of the universe. It is certainly true that the fundamental laws of science and mathematics, with their marvelous and intricate dependence one upon another are of more value to the individual than the principles of a language he will never speak. The laws of Mathematics and Science are God-made,

*Read before the Illinois High School Conference, November, 1921.

while the principles of language and history are due to man's intelligence.

A proper study of Mathematics assists in the development of character. The pupil soon learns that, to get correct results in mathematical processes, he must obey the laws of the subject which he has learned and must obey them implicitly. A failure to reach accurate conclusions is due to his own errors and nothing else. What this experience is worth to the individual in later years is difficult to estimate, but since nearly all the actual failures in life are due to the individual's mistakes and negligence and not to outside influences, it is well for him to encounter early that study in which he pays dearly for his own mistakes.

Furthermore, the difficult problems of Mathematics furnish abundant material to test the powers of the keenest minds, while the weaker student may at the same time find exercises within the range of his ability. One of the most valuable results from the study of mathematics is the acquiring of a good habit of preparing the day's lesson. The assignments of Algebra and Geometry are definite enough that even the student who has never learned how to study is soon able to prepare a portion of each lesson and to know positively that that much of the lesson has been completed. As he becomes more and more able to do all of the day's assignments, he develops a habit of study that will benefit him during his whole life.

But let us come nearer home to our subject and ask the definite question, "Why Study Geometry?" In what way does this subject aid in the development of the individual so that it is entitled to a place in the school curriculum? Is it more important than other subjects of Mathematics and Science that might be put in its place? Let us answer these questions from the standpoint of what has been said in the preceding paragraphs. Geometry in common with all mathematics deals with the fundamental facts of the universe and every person of culture and refinement should know something of these truths. The poetry has never been written, the music has never been played that contains one-tenth the appeal to the human soul that is found in the harmonious motion of planets, stars, and other heavenly bodies. The difficulty, however, is that many of the

beauties of creation are beyond the comprehension of the average mind on account of the inability to understand the mathematical principles on which these marvelous things depend.

I feel that I owe you an apology for using so much time on this part of my paper on matter that is not immediately connected with the topic assigned me, but I feel that I could not consistently discuss the topic of experimental Geometry without putting it on the same basis as any other branch of mathematics. Certainly we all want to feel that there is more worth in the subject than merely teaching a new method for drawing a right angle or for ripping a board into several equal strips.

There are three ways for the student to get possession of the truths of Geometry. The first method is by experiment, as, for example, he constructs several equilateral triangles and, after measuring the angles, he arrives at the conclusion that the value of each angle of an equilateral triangle is 60°. While such reasoning is not logically sound, yet we must admit that many of the facts of science have been discovered in just this way.

The second method is by experiment followed by formal proof. This way is the most logical of all. For example the student finds that the two diagonals of the rectangle that he has constructed are equal. After measuring the diagonal of other rectangles, he proceeds to prove that the diagonals of any rectangle are equal.

The third method is by logical proof without experimentation. The student works with given propositions to discover their proofs or he attempts to discover new truths by juggling logically known facts. I think we will agree that this third method is the most difficult of all. We have wasted much time in the past by beginning Geometry at this point. Even though the student has no experience with geometry till the tenth year, yet it would be wise to devote some time, even a half year, to the first two methods of acquiring mathematical truths, before we require formal proofs. This, however, is merely a makeshift to take care of those students who arrive at the tenth year without any knowledge of Geometry. The young student beginning the study of Geometry, in the latter years of the grades or the early years of the junior high school, should learn at first

a few definitions and then proceed to the use of the tools of the subject. The necessary vocabulary after the first definitions will be obtained gradually as the need arises. All of the work should be centered about the construction processes. He will need and should learn to use the following tools and materials:

A hardwood straight-edge divided into inches and centimeters; a pair of compasses; protractor; 30°, 60°, 90° right triangle; 45°, 45°, 90° right triangle; good drawing paper, cross-section paper, and good firm paper for experiments in paper folding, and a supply of yard-sticks, string, wire, thumb-tacks, etc.

The work may proceed as follows: After the student has become familiar with the use of the tools he should learn the simpler constructions with the ruler and compasses, such as bisecting a line segment, constructing a perpendicular to a line at any point on the line or through any point not on the line, bisecting an angle, and constructing an angle equal to a given angle. After these constructions are mastered and the pupil has tested his results in many ways he is ready to settle down to definite experimental study.

The work should all be done by means of the laboratory plan. He must be given little hint of what the results are to be but must find them out for himself. As, for example, one day's lesson is about the value of the angles of triangles. We will assume that he has already learned how to construct a triangle with the compasses and straight-edge when the lengths of the three sides are given.

The pupil will do the following exercises:

(a) Draw several triangles and measure the angles. Find the sum of the angles of each triangle. What is your conclusion?
(b) Construct several isosceles triangles and measure the angles. What relation do you find between any of the angles of an isosceles triangle?
(c) Construct an equilateral triangle and measure the angles. What is your conclusion.
(d) Darw several scalene triangles and measure the angles. Where is the largest angle in each of the scalene triangles?
(e) Can you find the sum of the two acute angles of a right triangle without measuring?

After the young pupil has done a considerable amount of experimental work, he can begin simple proofs in an informal way. The five well-known theorems relating to the congruence

of triangles should be done entirely by experiment. After the student is satisfied that they are true he can undertake such simple exercises as the following:

(a) ABC and XYZ are two congruent triangles, angle A equal to angle X, etc. The altitudes CD and ZK are drawn to the respective bases AB and XY. Draw the two figures and show the triangles ADC and XKZ congruent.
(b) Draw a line from the vertex of an isosceles triangle to the middle point of the base and show the two triangles congruent.
(c) AB and CD are two equal chords in a circle whose center is O. Draw OA, OB, OC and OD and show the two triangles to be congruent.

The amount of material available for experimental work in Geometry is so great that no class can be expected to do a great part of it. The teacher should make a careful selection from the following:

(a) Study of lines, angles, triangles, polygons and circles.
(b) Parallels.
(c) Drawing to scale and similar figures
(d) Accurate proportions with the bar graph and circle graph.
(e) Symmetry.
(f) Ratio and proportion.
(g) Constructing by paper folding perpendiculars, bisectors of angles, bisectors of lines, etc.
(h) Pythagorean theorem.

It does not seem wise to say positively where the material already discussed should be done. Certainly some of it should be done before the end of the eighth year. In schools where the mathematical program can be so arranged, a considerable amount of work in experimental Geometry can be done in the seventh year. Of course, if it is done in the seventh year, the work must be simplified to be within the range of ability of the seventh grade pupil. If experimental Geometry is neglected entirely till the tenth year, then by all means give a few weeks of it as an introduction to formal Geometry.

So far we have dodged the question of unified mathematics. Experimental geometry will be greatly enriched by teaching something of Algebra at the same time. Many of the formulas of Algebra can be illustrated so nicely with the rectangle, bar graph, etc., that it seems a shame if the pupil does not have this correlation of mathematics.

We will close this paper with a few suggested exercises for experimental geometry. It will be noticed that these exercises are in groups and the last one in the group, if possible, deals with the practical application. They are arranged in this way merely to add interest to the subject, and thereby to make it more attractive. At the same time let us use all the means in our power to convince the pupil that the subject of Geometry contains much more than the practical application.

Suggested Exercises.

1. Construct a rectangle and draw the diagonals. Measure the lengths of the four segments into which the two diagonals are divided at the point of intersection. Where is the center of the rectangle? How can you find the point in the center of the ceiling of a room where a light fixture should be placed?

2. Construct several perpendiculars to the same line at different points on the line. What do you notice about these perpendiculars? How can you use your right triangle to draw a number of parallel lines on a sheet of paper?

3. Make a quadrilateral out of four yard-sticks by fastening them together at the vertices with a single nail at each vertex. Can you change the shape of the quadrilateral? Is it always a parallelogram? A square? Measure the diagonals. Are they equal? Write several conclusions. Could you use this instrument to draw parallel lines on the blackboard?

4. Draw any triangle. Construct a second triangle with sides twice as long as the sides of the first triangle, and construct a third triangle with sides three times as long as the sides of the firsttriangle. What do you notice about the shape of the triangles? Measure the corresponding angles of the triangles. State your discovery as a theorem.

5. Draw a circle. Construct a central angle of 90°. Draw several inscribed angles intercepting the same arc that is intercepted by the central right angle. Measure the inscribed angles with the protractor. Repeat the process with a central angle of 120°. State your conclusion as a theorem.

6. Draw a rectangle. Draw a second rectangle upon a base equal to that of the first rectangle whose area is three times as great. How do their altitudes compare. The areas of rectangles having equal bases have what ratio?

7. Draw a triangle with one side twice as long as one of the other sides. Bisect the angle included by these sides. Measure the segments into which the bisector divides the opposite sides. What is their ratio? How does it compare with the ratio of the two sides of the angle bisected? State the fact in the form of a theorem.

8. Draw a triangle. Upon a base three times as long as the base of the first triangle draw a second triangle similar to the first. Draw the altittudes to the bases. How do the altitudes compare in length? Compare the areas. Tell how to construct a triangle similar to a given triangle with four times the area.

9. A man whose incomes is $3,000 per year spends $500 for rent, $800 for provisions, $300 for fuel and other household expenses, $600 for clothing, $200 for charity, $300 for luxuries, and save the remainder.

Construct a circle graph to show the various expenditudes. Construct bar graphs to show the same expenditures, using one square centimeter for each $100. Which do you consider the getter graph.

10. Draw any triangle. Construct and measure one altitude, measure the base to which the altitude is drawn, and compute the area of the triangle. Draw any quadrilateral. Draw a diagonal dividing it into two triangles and by the preceding method compute the area of the two triangles. Explain how you could find the area of a four sided lot that is not a rectangle.

11. The three sides of a triangular lot are 18, 20 and 24 rods. Draw a triangle to the scale of one centimeter to one rod and compute the area of the triangle and the lot.

12. Draw any polygon. Extend one side at each vertex. Beginning with one exterior angle rotate a pencil through all of the exterior angles. What is the amount of the rotation? Draw another polygon having a different number of sides. Repeat the experiment. What do you conclude about the sum of the exterior angles of a polygon?

HOW CAN I BRING THE SOUL OF MATHEMATICS TO MY PUPILS

By ALBERT H. HUNTINGTON
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To adequately discuss the subject, it seems necessary to define what one means by "the soul of mathematics," or at least to phrase it in another way more definitely suggestive. "Soul" means much to some and little to others of us. Among other definitions, Webster says that "the soul" is "the seat of real life, vitality, or action; the animating or essential part." I believe that this is what we wish to discuss with regard to the teaching of mathematics, so that we may phrase our subject, "How can I bring to my pupils the real vital part of mathematics, the animating, energizing part, the part that has made men devote their whole lives to its study and to wish that they had the more life to live?" The question is one that each one of us needs to answer carefully, thoughtfully, because we will all admit that there is a vast difference between the effects we wish to produce in our pupils and those we do produce.

Do we succeed as teachers if they fail as pupils? Dr. Kilpatrick says, "As teachers we fail if at any time, our children are not disposed and equipped to go on." Taken as they come in various classes, how large a percent of our pupils of mathematics would continue studying the subject if they had their own untrammelled way in the matter? And secondly, how many are equipped to go on? This is a pretty large indictment but there is yet another.

Flexner in "The Modern School" says "A large percentage of our pupils fail (in mathematics) and they take too long a time in doing it." This means that too many fail in mathematics and that it is too costly both to the pupils and the public to carry so many of the pupils so far before they find out that their aptitude and desires carry them into some other field of study.

Both these statements are as true indictments of the teaching of other subjects as they are of the teaching of mathematics, but I have chosen to use them here in order to present the case as vividly as possible. Are they true indictments? Yes, to a large

extent. "Disposed and equipped to go on" is a phrase rather carelessly coined so far as meaning and consequences of its literal interpretation go. We do not nor would we want all students to be specialists in mathematics; but we are justified in hoping that we can get all students to realize the permeations of mathematics in the everyday life about them, its importance, and their need of knowing enough of it so that in a given situation in which they are likely soon to find themselves, they will not be at a loss for the fundamentals at least to meet the measurable demands of the situations. We do not want them like a man who can not swim and perchance falls in a well where he has to fight the water to keep it from drowning him; rather we wish them like him who, being able to swim, uses the water to hold himself up until he makes a convenient toe hold, and using it, makes another and another until he finally gets out. We are justified also in hoping that some of our pupils will see the beauties of mathematics, catch the spirit of it, and devote themselves to it, being thereby the better enabled to serve themselves and their fellow men. But how can they do this if it is not first given to them "in spirit and in truth?"

Taking up the second indictment, what are the causes of the failure of pupils in mathematics? I believe that we may attribute a large part of the failure to the fact that the pupils do not begin the subject in the right state of mind, nor do they get the real, vitalizing, energizing spirit of mathematics in many classes. Let us examine the state of mind in which many go at the subject.

A child may hear his father or mother, particularly the mother, in conversation with the neighbors who have children inn school, say, "Oh! I never was any good in arithmetic when I went to school. I suppose that John will have trouble." Consequently, in accordance with the conclusions of Jung, Freud, and other psychoanalysts that the crises of life are to be found in the earliest years and that nothing comes into the mind but what has some connection with all the rest, John is prepared in his mind for failure when he does get in school. One day he does not prepare his lesson satisfactorily. When called to account for it he may truthfully say "I don't know." From letting the one slip, he may go from bad to worse, aided and

abetted by his environment at home; and then the excuse verily given becomes, "Well, mother always had a hard time in arithmetic and I suppose I am like her." The unfavorable environment may as well be created by an elder brother or sister, failing in algebra or in geometry, or by being put in a class at school taught by a teacher who has a reputation created by failing pupils. In cases like this, the environment in the home or circumstances at school have turned the child's mind in a direction from which sometimes only the most determined effort on the part of the teacher can reclaim it.

The next point I wish to emphasize is the value of enthusiasm. We have just come through a great war in which, toward the end, the one striking difference between the opposing soldiers was the presence of enthusiasm on the one side and the lack of it on the other. We all know which won, and we all realize in some way the value of enthusiasm to one in athletics, to one in doing his work, to one who is leading others, to one who is away from home. Enthusiasm in any class is an energizing spirit which if guided aright will help in surmounting great difficulties and make up for a multitude of sins. It is only when the sins are very many that we are justified in finding fault with a whole class because there is no other way quite so effective of killing enthusiasm as by finding fault habitually. We ought to be very sure that we ourselves are not guiding our class very poorly if we have to find fault with it publicly very often: most certainly if we wish to talk to individuals, we should do it individually. Summing thus far, if we wish our mathematics to appeal to the pupils, we should cultivate the right attitude on the part of the individual and class, and nourish and guide aright all enthusiasm on the part of either.

Next we need to become acquainted with our pupils, to find them out, to learn their interests. Learning their interests, we naturally appeal to them more, and direct appeal leads to an enthusiastic return. Individual enthusiasm is contagious, and leads to group enthusiasm and interest. Interest leads to effort if properly directed. When a class as a group is putting forth a collective effort to get hold of a new idea, something pleasing to them is bound to come of it, and the effort thus expended leads to new interest. To illustrate—a child may become greatly

interested in learning to count. Probably the younger he is, the more pleasure he will take in it. The ability seems to him to be an accomplishment over which he has mastery, giving him a sense of power over a little of the outside world. He compares his new mental state with his previous one and feels a real advance. It gives him pleasure, and therefore he uses it whenever he has opportunity. A little girl having one doll is given another and another. No one could be more proud nor derive more evident satisfaction than the little miss who can and does point to the evidence of it, "I have 'free' dolls." As the acquirement goes on, her number idea grows. For the city child, the number sense grows with counting boats, cars in a train, people in a group; with the country child, similar growth takes place in hunting eggs and telling mother how many there are, countinng the chickens, calves, pigs, and so on. The desire to excel spurs the child to more activity in counting. My own boy, turned loose on the farm, came in with the startling announcement, "Papa, I found 41 eggs today. They didn't find but 26 any day before I came here." This boy had applied his knowledge of counting to an actual situation in which he had found himself where he was in some way connected with breaking a record. It had given him great satisfaction because he could use his knowledge of counting in something that seemed eminently worth while.

For the elementary classes, this experience gives us the hint to teach that mathematics which fills the felt needs of our pupils. Many of us give examples 1-10 on page 63 of the text for tomorrow's lesson and let it go at that. At the start, man was not given a set of books containing knowledge arithmetical, algebraic, or geometrical. Instead of books he was given a more or less philosophical mind with which to think and many forms and groups provocative of thought. Quoting from Hill in "Geometry and Faith," "Many of the geometer's *a priori* laws were first suggested by the laws of nature. Natural symmetry leads us first to investigate the mathematical law which it embodies; then the mechanical law which embodies it." We can show our pupils that man in contemplating nature has thought out laws, then he has gone to nature to verify them and in the doing of it has been provoked to new discoveries resulting in

new conclusions and *vice versa.* Through alternations of discovery, proof, discovery, mathematics and natural sciences have grown to be the organized bodies of knowledge that they are today. In the same way as they have grown in the past, they are growing now, and will continue to grow in the future. I believe that nothing stirs a boy or a girl in quite the same way as to discover at the proper time, that mathematics is not a dead, finished subject, but that it is a live, growing one in which men are constantly making new discoveries and applications. In teaching, we can let him discover for himself many of these laws, and he will be filled with a sense of achievement in just the same way as the child learning to count.

This shows us also that we are prone to make too much use of text books in our teaching, and not enough of nature. We don't show the children how natural is the origin of much of our mathematics. How we may do this is shown in a most striking fashion in the "Illustrated Mathematical Talks by Pupils of The Lincoln School," pp. 25-31. The pupils presenting the material there found will always remember it, and they will be talking about it with each other, thus creating a real mathematical atmosphere and propaganda. I have taken a class out-of-doors in New York City with the subject "Triangles." We found them in the shapes of trees in the park, in the leaves, in places where three streets crossed, and saw them in structural work for whole buildings, for holding up scaffolding, fire escapes, keystones for arches, and for ornamentations. Coming home we built a meccano bridge to find out why triangles were so necessary in structural work. For a theme following, New York City was pretty thoroughly searched for triangles, physically, structurally, artistically, and physiologically. One boy said that the profile of a dog's head was a triangle, and drew a figure to prove it. Such a thing is a good thing to do right in the time of this class recitation which we so zealously guard against interruptions; for although the class made no progress in the text book that day, it acquired an impetus that made up several ordinary lessons, and created a spirit of discovery for things mathematical. If mathematics as we teach it pertains directly to the child's environment, helps him to understand it better, makes itself worth while in his work and in his play, the child will appreciate it and be filled with the spirit of it.

For the more advanced classes, say from the end of the ninth school year on, the students of mathematics should be animated by a purpose in their own soul to study the subject. Until this time boys and girls do much of their school work either to please their teacher or their parents, or take it as a matter of course. If they have succeeded in it as the school counts success, they have the added impulse of success and sense of accomplishment to spur them on to do more; if they have not, much or all school work is becoming irksome and they are looking for a substitute. I would not have pupils continue in mathematics beyond the ninth year unless they really liked the subject, unless they were going to college, or unless they were preparing for some particular vocation. Those giving the second reason ought to be made to feel that the colleges were reasonable in making the requirement, and those in the third group ought to see the applications of thir mathematics in the chosen vocation.

After a consideration of real things, let us turn to those of the imagination. The boy who drew the profile of a dog in a triangle was letting his imagination work, and I believe we can the more appeal to the imagination of a child after he has a thorough understanding of the real. With younger boys and girls, we can let this lead on to a discussion of larger things as compared with small, people on other planets, the distances of the stars, and can with propriety and patience answer to the best of our abilities the thousand and one questions that a lively class is sure to ask. In answering these it pays to be absolutely honest and if we don't know the answer to say so. Then we can together look it up and the teacher and class will derive a new pleasure in being co-workers with the same purpose. With these older classes such discussions may well deal with ideas of the infinite, of space or time, something of the nature of which Dr. Smith has indicated to us in the November number of the *Teachers College Record*. In my own classes that know physics, I have felt a preference for introducing here the location of the image of a luminous object found by a double convex lense and trace its movements as the distance of the light giving object from the lense is changed. A class can actually see a good deal of the change with the physical eye, imagination can easily picture the rest, and there is opportunity for so many thought-provoking questions.

Children and young people acquire a greater respect for mathematics if, as the class takes up new subjects or has trouble with old operations, the teacher gives little historical sketches of just how the race overcame this trouble or met that difficulty or tells why this particular process is in the course. All these should be spontaneous. Sometimes it is wise, though, to tell a committee about some historical reference or application, and ask them to look it up for the class. If we wanted to tie a boat very firmly to the wharf, we would run many ropes out preferably to snubbing posts on either side. So the more connections we make between our mathematics in all directions with the life of the past, present, and future, the more do we make its permeations enmesh and enthrall our pupils.

Boys like to make things involving mathematical principles. One of the most instructive things we have had in an intuitive geometry class has been a meccanno set, from which all sorts of forms can be built, and in which imagination can be allowed free play. Boys like to play games, also, in which the use of positive and negative members, the fundamental operations, and other processes, are stressed. It is well to let them get a good deal of drill this way, because mere mechanical drill kills enthusiasm and spontaneity of ideas, two very important parts of good class spirit. Speaking in psychological terms, we may say that the purpose of a child's drill is to reduce the fundamental operations to the plane of habit and using them in a large number of situations they must function as habits, leaving the child's mind free to do the rational thinking required to meet any new situation with which he is confronted. Games furnish a way, not enough used, of giving drill, and they also furnish the teacher an opportunity to step into the background and let the natural leaders of the class come forward.

And now a word for us as teachers. We stand before our classes as living embodiments of the subjects we try to teach. Are we fair in our dealings with our students? Then they would like to be in our classes, no matter what we teach. Do we explain so that with a reasonable amount of effort they can get it? Then they can understand and accomplishment leads on. Do we ever get enthusiastic? Then so will they. Do we ever crack a real joke? Then there must be some likable points

somewhere in the stuff. Do we connect mathematics with real men, past, present, with ourselves and the things about us? Then we aren't trying to sell them anything that we ourselves do not buy. Finally, do we ever forget all about our mathematics and disport ourselves as boys or girls or as men or women? Then exposure to our mathematics won't hurt them for a time at least. Is not this way the way most boys and girls size up our subjects and ourselves?

In summing up, the points I have made are these: To bring to our pupils the real, vitalizing, energizing spirit of mathematics, we must cultivate a whole-hearted enthusiasm on the part of the children and of ourselves; we need to present our subject to them not as a cut and dried one but as a living, growing one, permeating all our environment, necessary to a real understanding of and to progress in the life about us, furnishing them and us with real problems that will try our mettle but not too hardly; for man has been meeting them for thousands of years and solving them, yet the wonder of it all is that he has been making the most use of his mathematics in the progress of the last few years. Someone has said, "The true greatness of a work is in the thought which it embodies." We may paraphrase it—"The true worth of our teaching is in the thought which it provokes."

PAPERS BY PUPILS OF THE PLANE GEOMETRY CLASSES OF FULLERTON UNION HIGH SCHOOL

By LENA E. REYNOLD
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Toward the end of last school year nearly every member of our plane geometry classes undertook, in addition to his regular class work, the preparation of a paper on any subject of interest to him. The subject, chosen without reference to mathematics, was investigated to see whether or not it was in any way dependent upon, or connected with, mathematics. This plan originated with the pupils of the classes—probably as a reaction to the class work, which was designed to stimulate thought along that line.

The papers contained nothing new and startling—they added nothing to the total knowledge of the world; but they did give a considerable amount of knowledge, and much interest, to the students. The time devoted to the papers did not seem to detract from the quality of the regular class work, for in classes aggregating about one hundred there was but one failure, and many exceptionally good records.

The list of topics which were selected by the pupils comprise: Architecture, Art, Astronomy, Blood Enumeration, Bridge Construction, The Chemical Engineer, Commerce, The Construction of an Automobile, Dam Building, The Development of Mathematical Symbolism, The Engineer in the War, Geometry in Primitive Art, Headlights for Automobiles, Higher Mathematics in the Business World, Irrigating Systems, Life Insurance, The Los Angeles Aqueduct, Mathematics in General, Mining, Music and Mathematics, Navigation, Oil Production, Petroleum, Plumbing, Proportion in Everyday Affairs, The Pythagoreans, Railroad Surveying, The Refining of Petroleum, Road Building, and A Trip to an Airplane Factory.

To the mature person most of these topics obviously involve mathematics, but to the pupils this was, in many cases, not at all evident. It will be noticed that no attempt was made to confine the subject to geometry; one very wholesome feature of the project was that students had forced upon their attention the fact that geometry is but one branch of a very large subject.

To give the reader an idea of the quality of these papers we print herewith the one on Navigation and the one on A Trip to an Airplane Factory.

Navigation

By Douglas McGill

Navigation,* which affords a knowledge necessary to conduct a ship from place to place and by which a mariner may determine his ship's location at any given moment, is divided into two branches, namely: piloting and nautical astronomy.

Piloting is the most important part of navigation and requires the most skill, vast experience, the best of judgment, and wits always on the alert. In the navigation of the high seas, where nautical astronomy is employed, if an error is made in directing the ship's course, it is an easy matter to correct it by a later observation. But in the branch of navigation known as piloting, if an error is made, the result is frequently a disaster, since it is employed upon the approach to land. A ship, the navigator is taught, is usually safe on the high seas, and danger threatens upon the approach to land.

As his ship draws near land the pilot has ready the very latest charts, and particularly a large "scale" map, of the locality, showing shoals, lighthouses, buoys, etc. On this he finds the position of his ship. When conditions are fair his work is not hard, but should bad weather exist or low fogs menace, his work is of the most serious nature and his nerves are frequently racked at the responsibility with which he is burdened.

The navigator, when in sight of objects whose position is shown on the charts, may locate his ship's position by any one of the following methods:

A—Cross bearing of two known objects.

B—Bearing and distance of a known object.

C—Bearing of a known object and the angle between two known objects.

*My sources of information were:

(1) American Practical Navigator, by Bowditch. Published by U. S. Hydrographic Office, under the authority of the Secretary of the Navy.

(2) Navigation, by Harold Jacoby. Published in New York by Macmillan Co.

(3) Mr. Olsen, who was formerly a captain on Norwegian ships.

(4) Mr. H. Lindville, the First Mate on the ship Lily, which plies between Mexican waters and San Pedro.

D—Two bearings of a known object separated by an interval of time with the distance during that interval.

E—Angles between three known objects.

Frequently, in the application of these methods, Trigonometry is employed. There are many cases, however, where Plane Geometry is used.

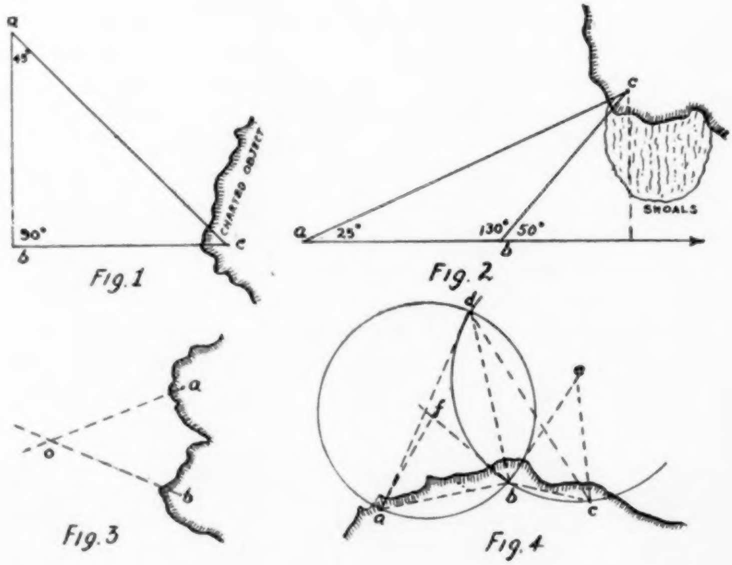
Where it is possible to get a bearing on a charted object the pilot has a valuable asset and a thing tangible with which to deal. It gives him the information that his ship is somewhere on that line, but the thing for him to know is, how far off he is from the charted object, for his chart shows him how far he must keep from that object to be in the safety zone. He employs, then, what is popularly known among mariners as the "bow and beam bearing," or the method as listed under *D*. An instrument called a "patent log" is towed at the stern, turning as it pulls through the water, registers automatically on deck the number of nautical miles traveled. This is read at the moment the charted object is 45° on the bow. (Fig. 1.) The same course is held until the object is directly abeam. This makes an angle of 90°, thus leaving the third angle $ACB = 45°$ or $180° - (90° + 45°)$. It is a geometrical fact that sides opposite equal angles are equal. Therefore, the distance the patent log registers will show how far the ship is from the charted object, for it registers the distance AB, and that is equal to BC. After consulting the chart, the pilot knows how near the danger zone he is taking his ship.

When it is known that a reef or shoals lie off the charted object, another method is used, since it may be seen that it might be extremely dangerous to wait until the ship is abeam before taking a bearing. This method is known as "doubling the angle on the bow," and is worked thus:—

A bearing is taken when the ship is, for example, 25° (Fig. 2) on the bow, and the log is read at the same time. The same course is held until the angle on the bow is 50°, or double the first angle. Here the log is again read. The angle ABC will be $180° - 50° = 130°$, and the angle ACB will be $180° - (130° - 25°) = 25°$. Therefore, the distance $AB = BC$ (sides opposite equal angles are equal), and the distance BC is the distance off the object when the second bearing was taken. In

case a strong current is running, the pilot must make allowance, otherwise the distance off the charted object would be in error.

A bearing of a known object and the angle between that and another known object is not often used, as it is preferable to take a cross bearing.



A cross bearing of two known objects is determined thus: choose two objects which show unmistakably on the chart, observing the bearing, which each makes with the ship (Fig. 3). Then draw on the chart lines which make these angles and pass through the object *A* and *B*. The ship will be on some point on each line. Therefore, it must be on the intersection *O*, since that is the only point common to each line. Thus the pilot finding his ship located at a point *O* knows, from his chart, whether it is in safety or danger and holds his course or changes it as the case may be.

At times a third object is discernible. Here the same method is used, between the third object and one of the others. If, then, the point of intersection falls upon the point *O*, the pilot has checked the accuracy of his first "find."

The "Three Point" problem is much used among mariners. It is to find a point, such that three lines drawn from it to three given points shall form given angles with each other. The point to be located on the chart being the ship's location, A, B and C are the three charted objects along the coast. (Fig. 4.) The angle $C D B$ is observed to be 20° and the angle $A D B$ is 40°. Since the angle $B D C$ is 20° an angle at the center of a circle intercepted by the same chord would be double or 40°, and the angles formed by the radii of such a circle to the extremities of that chord would be $\frac{1}{2}(180° - 40°)$ or 70° each. Therefore, construct angles $E B C$ and $E C B = 70°$ each. With a radius $E B$ describe a circle. In the same manner construct angles equal to 50° each on the extremities of the line $A B$, then with a radius equal to $F B$ describe a circle intersecting the first drawn circle at D, which point located on the chart gives the correct position of the ship.

In navigation along a coast where many sunken rocks or shoals menace a ship there is a method used whereby a ship may pass through in safety between those charted ogres of the sea. This method is known among seafarers as the "Danger Angle." For example: Let S and S' be two sunken reefs with A and B two charted objects on shore (Fig. 5). In order to avoid the reef S', draw on the chart about the reef a circle, using as a radius the distance from the center of the reef to the point inside which the ship may not enter in safety. Construct, then, another circle which will pass through A and B and be tangent to the first circle at E. Draw the angle AEB.

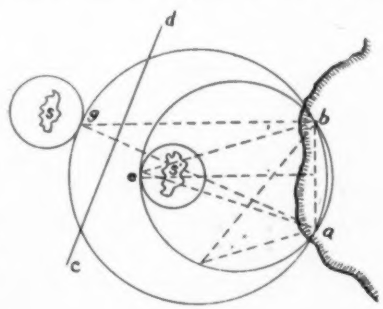


Fig. 5

It being a geometrical fact that the chord $A B$ will subtend equal angles at any given point on the circle, and greater angles at any point inside the circle, it will be seen, that so long as the ship passes on a course, such that the angle between it and the objects A and B never becomes greater than angle $A E B$, it will be safe so far as the reef, designated as S', is concerned.

To avoid the reef S, draw, as was done previously, a circle about S which will show how far from S the real danger lies. (This, of course, being shown on the Coast Pilot.) Then describe another circle which will pass through A and B and be tangent to the circle which lies about the reef S, at the point G. Measure angle $A\,G\,B$, and as long as the chord $A\,B$ subtends an angle greater than $A\,G\,B$ the ship will be within the circle $A\,G\,B$ and will pass between the dangers S and S'.

Thus it may be seen that upon the approach of land many problems present themselves. The pilot must, of necessity, make himself extremely familiar with his charts, must always have his instruments in perfect working order, but above all he must have an alert mind. He must never for a moment allow himself to be in doubt as to his ship's location and he must always check up his location as land marks appear. Whether it is passengers or freight which his ship carries, or whether it is only a ship traveling unladen, his responsibility is most grave, and it behooves him to know always that his observations are absolutely correct and that he is certain of the correctness of the solution of the navigation problems as they arise.

A Trip to an Airplane Factory

By William McBride

Aviation is the topic of the day. It is the ideal means of fast, comfortable and safe traveling. It has not yet arrived at perfection, but like the automobile it is beginning to come to the front as an economical means of conveyance.

Even though many of the people of the United States have ridden in an airplane, yet a very small percentage know the whys and hows of an airplane.

Let us pretend that we can take a trip through an airplane factory. We must also suppose that we own an airplane which will take us to the nearest factory in a short time.

First, we must get a pass which enables us to go through the buildings and grounds. A gentleman, who has seen service in the aviation department overseas, volunteers to show us through the plant. He leads us toward a large building some distance away. On the way there he explains to us the theory of an airplane.

"The airplane," he begins, "is not such an intricate piece of mechanism as one might imagine. The air is rushing under the wing, which, as you know, is slightly tipped, and a partial vacuum is formed on top of the wing. Then the reaction between this vacuum and the force of the air rushing underneath the wings tends to support the wing. This wing, or panel, is made large enough to support the whole body of the airplane. All this ratio between the power, the slant, or angle of incidence, of the wings, the wind resistance, the weight of the machinery, etc., must be carefully calculated by highly trained mathematicians, who make this their business.

"The propeller is to the airplane what it is to the boat, and it acts on the same principle, namely, that of pushing in the case of a pusher, or pulling in the case of a tractor, its way through the air. It is this power which pulls the airplane along fast enough to keep it in the air.

"Underneath the airplane, you see two small wheels, and near the tail a small peg." He is pointing to a training machine which is circling above our heads. "This is the landing gear. The wheels are rubber tired and are mounted on shock absorbers, which, as you see, are part of the construction which is fastened to the body, or fuselage, directly under the lower wing. These are to take up the shock of landing. The peg at the rear of the machine serves as a brake and as a support to keep the tail off the ground."

We are now entering a building in which the manufacture of propellors is progressing.

We are surprised to see so many men in the building, since we imagined that everything would be done by machinery. Our guide looks at us and smiles.

"Yes, they are all made by hand," he says.

"Here is where the propellor is shaped," he continues, pausing by a buzzing band saw. "Each layer of wood is sawed into shape right here. Then the pieces are taken into a glue room, heated and glued together. They dry slowly, and then they are smoothed into shape by drawing knives, or in some cases by a routing machine. After they have been smoothed sufficiently they are tested for pitch. Pitch is the distance a propellor would screw through the air were the air solid and were

there no slip. Then the 'slip' of the propellor must be reckoned with. That is, if the propellor had a pitch of ten feet it might advance the machine only seven feet. This would mean a slip of three feet."

Our guide tells us about the difference in the shapes of different propellors which are being, and have been, used.

Soon, we come to a room full of queer looking apparatus. There are odd derricks, scales, levels and all manner of apparatus. Our guide informs us that this is the testing room. Here is where propellors are accepted or discarded.

Then we pass on to another building where the construction of the wings is progressing.

"The wings," our guide informs us, "are the main parts of an airplane. They are the supporting surfaces. Great care must be taken to get them smooth and entirely free from bumps or blemishes in construction. Then, to insure smoothness, they are coated with 'dope.'

"The fuselage, or body, is made of a canvas covered skeleton of black walnut and spruce woods. It is so built that it has a stream line effect. This means that it is shaped like a fish."

Then we see a part of the building where a completed fuselage is sitting, and on it mechanics are working to determine the "angle of incidence" and the "stagger."

Our friend, the guide, points this out as we stand there watching. He probably notices our perplexed look, and explains his terms.

"Stagger," he says, "is the horizontal distance from the front tip of one wing to the corresponding tip of the other. You see," he said, pointing to a finished machine which could be seen through the open door, "one plane is set back from the other. Well, this distance which it is set back is the stagger. This stagger can be computed by the weight of the machine, for if the stagger is too great the machine will be nose heavy, and if too little, tail heavy.

"Now for the angle of incidence. See that plane out there? You notice that each wing is tipped somewhat. This is the angle of incidence. Obviously, it is for the purpose of lifting the machine off the ground. Also, you can see that the greater the angle, the less speed. That plane which you see out there

has two degrees, a hydro-airplane four, and so on according to the size and power of the machine.

"It is getting late," he remarks, pulling out his watch, "but we can see some more buildings and what they contain."

We pass hurriedly through a few more buildings, and in every case we see many people hard at work, but there is one thing which we notice especially. That is the number of mathematicians. In every building is a force of well trained mathematicians. Many branches of mathematics are used. Here we see a man calculating the distance a certain machine can travel on so much gasoline. He is using arithmetic and algebra. Over there is a young man drawing plans for airplanes. Everything must be carefully calculated so as to get the proper ratio between lift, drift and power. Our guide tells us that these men—most of them young men—are drawing excellent salaries.

We think that since we are there we might as well take a look at the hangars, or airplane garages, on the way out, so we walk over that way.

We walk around the field, gazing at the planes and hangars. While we are so engaged, our guide tells us of some of the problems which an aviator runs up against. The talk drifts to cross country flying. I ask our friend how an aviator can tell whether he can go so far and return, or whether his gasoline would give out.

"Well, here is a little illustration," he starts out. "I shall show you this problem in a way which will be easy to understand. There is a shorter way, but it seems rather complicated, unless it can be seen down on paper. I shall show you in a way which is a little longer. When I was in training camp, it was the custom, and is still, for a student to take a solo cross country flight. I had to take it, and so did the rest.

"When my turn came, there was quite a wind blowing. I had to calculate how far I could go and return on three and one-half hours' gas. This is how I did it. First, by means of an instrument which I shall not try to describe I found the wind to be blowing at the rate of forty miles an hour. One must always allow a half hour of gasoline for climbing and for margin. This leaves three hours, which at seventy-five miles an hour is two hundred twenty-five miles, or one hundred twelve

miles out and one hundred twelve miles back. As I was to go directly against the wind, the radius of my flight was altered as follows: My speed outward was seventy-five minus forty miles an hour, or thirty-five miles per hour, and the speed on the return trip, seventy-five plus forty miles an hour, or one hundred and fifteen miles an hour, or three and twenty-nine hundredths times as fast and occupying a time which I shall call X. The time on the outward trip would then be three and twenty-nine hundredths X, a total time of three and twenty-nine hundredths X plus X, or one hundred eighty minutes. By algebra, we find that X is equal to forty-two minutes for the return trip and one hundred thirty-eight minutes as the outward. The distance covered on the outward trip is then one hundred thirty-eight sixtieths of thirty-five, or eighty and five-tenths miles. This is the radius of flight.

"Oh, yes," he remarks, noting the dazed expression on our faces, "you'll need mathematics if you expect to be aviators. In fact," he says, "if some boys hadn't studied mathematics when they were in school we would not have the airplane today."

We have now reached the place where we entered. Our friend shakes our hands warmly as we leave, and, after we have thanked him, he offers to help us in any way possible if we should decide to become aviators.

DISCUSSION

A New Type of Examination. Just now serious thought is being given to the general intelligence style of examinations in academic subjects and a commission has been appointed by the College Entrance Board to investigate the growing interest in this style of test.

While the chief desire is to reach a more uniform and fair marking system, there is another aspect that perhaps has been overlooked, namely the saving of time in correcting papers. For the past year and a half I have been giving my mathematics classes the so-called plus and minus tests and two noticeable facts have been repeatedly brought out. The first is that the results give in every case just about your personal estimate of the pupil's worth—there are no "surprises." Secondly, while a somewhat longer period is required to make out the examination, the correcting time is materially reduced. This is especially true with large classes. A master copy is placed beside each paper and it is the work of only a few moments to scan down the blanks and check off inaccurate answers.

The number of questions of course will vary with the time of examination and the nature of the subject, but for an hour and a half test sixty-or seventy questions can be given. The papers are run off on a mimeograph and a sheet given to each pupil. All computations are made on another sheet of paper which is collected at the end of the period with the examination sheet.

I have arranged a few specimen questions below:

Part I

In the following place a plus sign in blank if you think it true and a minus sign if false.

1. $x + x + x = x^3$ ....
2. $(a + b)^2 = a^2 + b^2$ ....
3. $(r + s)(r - s) = r^2 - s^2$ ....
4. Eight greater than $4x$ is $8x + 4$ ....
5. k oranges cost 50 cents. One orange costs k times 50 cents. ....
6. If $2x = 1 - x$ then $x \doteq \frac{1}{2}$ ....

7. $m + 2[m - (m + 2m)] = 0$ ....

8. $(a^{-1}b^2)^{1/2} = b\sqrt{a}$ ....

Part II

Fill in the following blanks with correct results.

1. Solve equation for y. $2y - \frac{y}{2} + 5 = y$ ....
2. In formula $F = \frac{5}{9}C + 32$ find value of C. ....
3. In how many places will the graphs of $x + y = 5$ and $xy = 8$ intersect? ....
4. Find third term of $(x + y)^5$ expanded. ....

Part III

In the following questions place a cross in blank beside the expressions which you consider simplified.

1. $\dfrac{a + b}{\sqrt{2}}$ ....
2. $2\sqrt{3} + \sqrt{4}$ ....
3. $2a + \frac{b}{c} + d$ ....
4. $\dfrac{m + n}{\frac{1}{2}k}$ ....

Part IV

In the following problems underline the statement in which the error occurs.

Solve $\begin{cases} x^2 + y^2 = 5 \\ x - y = 1 \end{cases}$

1. $x = (1 + y)$
2. $(1 + y)^2 + y^2 = 5$
3. $y^2 - y + 2 = 0$
4. $y = -2$ or 1

Part V

Fill in blanks with necessary data.

Given the equation $x^5 + 3x^3 - 2x^2 - x + 11 = 0$. This is an equation of the........................degree. The maximum number of positive roots is........................and the maximum number of negative roots is........................ This equation has........................ imaginary roots and........................real roots.

L. F. Babcock,
Riverdale School, New York City.

Formula for Conjunction Problems. In solving conjunction, clock and race track problems, I have been using a formula which, to my mind, simplifies the work significantly.

I derived the formula in the following way:

Let a = the time of one revolution of the slower body
b = the time of one revolution of the faster
and t = the total time consumed.

The first travels $\frac{t}{a}$ revolutions, while the second travels $\frac{t}{b}$ revolutions. The equation representing the time in which the faster gains a lap on the slower reads,

$$\frac{t}{a} + 1 = \frac{t}{b}$$

Solving for t, we get

$$t = \frac{a - b}{ab} \times \frac{ab}{a - b}$$

For example, Mercury makes a circuit around the sun in 3 months and Venus in $7\frac{1}{2}$ months. Staring in conjunction, how long before they will again be in this position?*

Using the formula:

$$\frac{ab}{a - b}, \text{ we have } \frac{7.5 \times 3}{7.5 - 3} = \frac{22.5}{4.5} = 5 \text{ months.}$$

Again: When, after 12 o'clock, is the first time the hands of a clock are in conjunction?

$$\frac{12 \times 1}{12 - 1} = 1\frac{1}{11} \text{ hours} = 1{:}05\frac{5}{11}$$

Taking a race track problem (Slaught and Lennes): Two automobiles are racing on a circular track. One makes the circuit in 31 minutes and the other in $38\frac{1}{2}$ minutes. In what time will the faster machine gain a lap on the slower?

$$\frac{38.5 \times 31}{38.5 - 31} = \frac{1193.5}{7.5} = 159\frac{2}{15} \text{ min.}$$

PLEASANT HUFFMAN, JR.
Hutsonville (Illinois) Township High School.

*Problem taken from Slaught and Lennes.

NEW BOOKS

The Place of the Elementary Calculus in the Senior High School Mathematics; and Suggestions for a Modern Presentation of the Subject. By NOAH BRYAN ROSENBERGER, Ph.D. Columbia University. Contributions to Education No. 117. vii. 81 pp. Columbia University, New York, 1921.

This little book is a carefully studied argument, well attested by facts, for the teaching of elementary calculus in the last high school year—in the twelfth grade. Perhaps the author would say it is a demonstration rather than an argument. The style is what Bertrand Russell would call apodeictic.

First it is proved that our schools are behind the schools of many other countries in mathematical achievement; then that reforms already begun here leave room where a new subject can be put; and finally, that the calculus is important enough to be put there. This much occupies nineteen pages, and is very much to the point.

Then the little book takes up the question of presentation. First, on the ground that "in many ways the mental development of the child follows the mental development of the race"; next, on the precedents estabished by recent text-books intended for students of this age or not much older; and finally, on the ground that the brighter of our pupils need more vigorous food, and that this particular form of educational activity can fulfil the essential condition of forming habits that the student will desire to maintain and strengthen throughout his lifetime,—on these three bases the author founds his presentation, which is indicated by the following order of topics:

1. The derivative. 2. The slope of the tangent. 3. Maxima and minima. 4. The differential. 5. Integration. 6. The definite integral (under topics 5 and 6 is included the area under a curve). 7. The length of an arc. 8. The volumes of certain solids (cylinder and cone—of revolution—and sphere).

The problems given to illustrate the author's method of presentation are bravely called practical; the reader must be generous. There is a little haziness about the astronomical information on page 56, and about the printer's problem on page 67.

The Labrador missionary of page 58 must have a very long straight beach, and make a lightning change on and off with his skiis. The graphs and integrals are all parabolas. The solids considered in finding volumes are the cylinder and cone of revolution and the sphere.

Too much definiteness should not be expected of a book that calls itself "a suggestion." On the other hand, it might well stimulate other suggestions. For one, is it wholly desirable to omit entirely the idea of a limit from the teaching preceding the calculus? It has been badly done, to be sure,—too early, and in an unnecessarily complicated way. Bad as it was, it seems now about to be wholly replaced by a method compounded of conjecture and hope. Without previous use of the limit idea, we run into difficulties; for instance, at the foot of page 78, our author applies the formula for $\int ds$ to the arc of a circle, obtaining as a result the value upon which his data depend. Applying the historical criterion, the strict evaluation of π might be put before the calculus.

Again, if we are to find warrant in history for the development of our teaching, we might take account of the period of Kepler, Cavalieri, and John Wallis. The experience of our pupils in graphs ought to make it possible to explain to them a "curve of sections" of any solid, to "prove" to them that the area of this curve of sections is numerically equal to the volume of the solid, and thus to get Cavalieri's Theorem in as a basis for solid geometry mensuration.

Turning to the author's account of the eleventh school year, we find maxima and minima by means of graphs, and some of the simpler properties of conics. Why not teach the derivative there? Certainly the practice of foreign schools should encourage us.

If we can scatter some of the fundamental ideas and devices of the calculus in the preceding work of the pupil, we shall avoid the feature which made the "Advanced Algebra" of previous years distasteful—namely, its heterogeneous character. "It is important that the student should feel that he is doing something," not merely getting ready.

George W. Evans,
Charleston High School, Boston.

Our Little Crusader Cousin of Long Ago. By EVALEEN STEIN. Boston. The Page Co. Pp. 144.

The Sieve or Revelations of the Man Mill Being the Truth About American Immigration. By FERI FELIZ WEISS. Boston. The Page Co. Pp. 307.

Smiling Pass. By ELIOT H. ROBINSON. Boston. The Page Co. Pp. 389.

The Triumph of Virginia Dare. By JOHN FRANCIS, JR. Boston. The Page Co. Pp. 357.

Marjory's House Party. By ALICE E. ALLEN. Boston. The Page Co. Pp. 315.

Famous Leaders of Industry. By EDWIN WILDMAN. Boston. The Page Co. Pp. 339.

(1) **Notes l'Équation de Fredholm.** By B. HOSTINSKY.

(2) **Notes sur les Quadriques de Révolution Qui Passent par des Points Donnés.** By DR. LADISLAV SEIFERT.

(3) **Les Quadratiques de Moutard.** By DR. EDWARD CECH.

(4) **Géométrie Projective de Cinq Droites Infinement Voisines.** By DR. EDWARD CECH.

These are publications of the Faculty of Sciences of the University of Masark.

Mathematics for Electrical Students. By H. M. KEAL and C. J. LEONARD, New York. John Wiley & Sons. Pp. 230.

Mathematics for Shop and Drawing Students. By H. M. KEAL and C. J. LEONARD, New York. John Wiley & Sons. Pp. 213.

Preparatory Mathematics for Use in Technical Schools. By H. B. RAY and A. V. DOUB, New York. John Wiley & Sons. Pp. 68.

Plane Trigonometry. By ARNOLD DRESDEN, New York. John Wiley & Sons. Pp. 110.

The Elements of High School Mathematics. By JOHN B. HAMILTON and HERBERT E. BUCHANAN. Edited by George William Myers. Scottt, Foresman and Company, Chicago and New York.

This text contains five chapters of arithmetical materials, one chapter which deals with intuitive geometry, and ten chapters which treat elementary algebra. The authors believe that "the burden of proof is now on the *stand-patter* who would maintain unchanged the existing order in high school mathematics."

The authors have not attempted to fuse or blend the separate subjects into a single composite type of course. The inclsion of a considerable body of arithmetic, and the simple, rational development of the algebra, are striking features of the text.

Classroom Drill Cards in Algebra. By JOHN DE Q. BRIGGS. Scott, Foresman and Company, Chicago.

A set of 130 drill cards covering the topics of a conventional course in algebra.

First Course in the Theory of Equations. By Professor LEONARD EUGENE DICKSON. John Wiley & Sons, New York and London, 1922. Pp. 168.

NEWS NOTES

A notable gift for the promotion of mathematical interests in America was announced by the Mathematical Association of America at its summer meeting in September, 1921. Mrs. Mary Hegeler Carus, as trustee for the Edward C. Hegeler Trust Fund, donated to the Association the sum of $1200 annually for five years for the purpose of publishing a series of monographs whose purpose should be to popularize mathematics by making accessible at nominal cost the best thoughts and keenest researches in this field set forth in expository form comprehensible to teachers and students of mathematics and to other readers of mathematical intelligence. The deed of gift includes the promise to capitalize this annual income by a permanent endowment fund if at the end of five years the project shall have proved successful. The trustees of the Association are taking steps to select a board of editors who shall have charge of all details connected with the publication of the proposed monographs.

An error appeared in Professor H. B. Roe's paper on "Minimum Mathematical Requirements in Agriculture" (January issue, page 33), in which the area of a circle was alleged to be 3 times the diameter, rather that 3 times the square of the radius.—THE EDITOR..

THE REORGANIZATION OF MATHEMATICS IN SECONDARY EDUCATION

The Final Report of the National Committee on Mathematical Requirements—To be distributed free of charge to all interested in securing a copy.

The complete report of the National Committee on Mathematical Requirements is in the press and will, it is hoped, be ready for distribution in April. It is published under the title "The Reorganization of Mathematics in Secondary Education" and will constitute a volume of about 500 pages. The table of contents given below indicates its general character.

Through the generosity of the General Education Board the National Committee is in a position to distribute large numbers of this report free of charge. It is hoped that the funds avail-